The Effect of Eccentricity on the Terminal Velocity of the Cylinder in a Falling Cylinder Viscometer

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The theoretical equations for the flow of Newtonian fluids in a falling cylinder viscometer with eccentricity are developed. The error in fall velocity of the cylinder caused by eccentricity is determined and presented in tabular form. An approximate method to determine the error in fall velocity caused by eccentricity for non-Newtonian fluids of the power law type is also presented. For the Newtonian case, $n=1$, the approximate method is shown to give excellent agreement with the exact method.

Laminar flow in the falling cylinder viscometer has been analyzed for both Newtonian and non-Newtonian cases: Lohrenz and co-workers (*9*) studied the falling cylinder viscometer for Newtonian fluids; Ashare et al. (*1*) presented approximate solutions for the flow of non-Newtonian fluids of the power law and Ellis model types; and Eichstadt and Swift (*4*) solved the theoretical equations for flow of Bingham plastic and power law fluids in the falling cylinder viscometer. Moreover, Lescarboura et al. (*7*) have presented a method for analyzing rheological data of time-independent non-Newtonian fluids in the falling cylinder viscometer without assuming a particular rheological model.

In all the studies mentioned, the assumption is made of concentricity of the falling cylinder in the fall tube and errors in fall velocity due to eccentricity are not considered. Studies have been made, however, of flow in eccentric annuli: Caldwell (*2*), Heyda (*5*), Piercy et al. (*10*), and Snyder and Goldstein (*11*) derived and solved the equations for laminar flow of Newtonian fluids in eccentric annuli, and Vaughn (*12*) analyzed the laminar flow of power law non-Newtonian fluids in eccentric annuli. The work done on the problem of eccentricity in annuli shows that large errors in flow rate may result when eccentric annuli are analyzed with the equations developed for concentric annuli. One may reasonably expect that eccentricity in the falling cylinder viscometer causes the same type of error as it causes in annuli. If the cylinder is free to move radially as well as longitudinally, a special problem appears when the cylinder is eccentrically located in the fall tube: The shear stress at the wall of the cylinder τ_w is not constant but varies with angular position, creating a force couple which tends to tilt the cylinder. At the same time, forces appear which tend to center the cylinder in the fall tube (*8*). These forces have been observed experimentally (*6*) to cause an oscillation which is damped by the fluid, presumably leaving the cylinder vertical and centered if the fall tube was made long enough. In viscometers built to date, to keep the length of the tube within reason, the cylinder is kept centered and vertical with pins or fins (*6, 9*). Some eccentricity is always present, however, due to machining tolerances. It is therefore important to know the magnitude of the error caused by eccentricity in a falling cylinder viscometer to determine the machine tolerances which will give results within a desired error bound.

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We report in this paper:

1. The development of the theoretical equations for Newtonian fluid flow in a falling cylinder viscometer with eccentricity.
2. The solution of the equations so that, given the eccentricity ratio, the ratio of terminal velocity of the eccentric case to that of the concentric case can be calculated.
3. The development of an approximate method to calculate the terminal velocity ratio as a function of eccentricity for power law non-Newtonian fluids in eccentric falling cylinders.

BIPOLAR COORDINATES

Because of asymmetry of the geometry, as shown in Figure 1, bipolar coordinates must be used to treat the case of an eccentric falling cylinder viscometer. The details of the transformation can be found in the literature (*3, 5, 10, 11*). The bipolar coordinates (ξ, η) are defined by the transformation

$$x + iy = ic\cot[(\xi + i\eta)/2] \tag{1}$$

where c is a constant and $i = \sqrt{-1}$. From Equation (1), one gets the following relationships:

$$e^{2\eta} = [(x+c)^2 + y^2]/[(x-c)^2 + y^2] \tag{2}$$

$$\xi = \tan^{-1}[(2cy)/(x^2 + y^2 - c^2)] \tag{3}$$

$$(x - c\coth\eta)^2 + y^2 = c^2/\sinh^2\eta \tag{4}$$

$$x = (c\sinh\eta)/(\cosh\eta - \cos\xi) \tag{5}$$

$$y = (c\sin\xi)/(\cosh\eta - \cos\xi) \tag{6}$$

Equation (4) shows that lines of constant η represent circles in the complex plane with centers at $(c\coth\eta, 0)$ and radii $c/\sinh\eta$ (see Figure 2). The inner and outer surfaces of the annulus are thus represented by lines of constant η which will be designated as α and β. With this

TABLE 1. TERMINAL VELOCITY RATIO Ψ_e AS A FUNCTION OF κ AND ϕ

$\kappa \downarrow$ / $\phi \rightarrow$	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.80	1.0037	1.0147	1.0331	1.0588	1.0919	1.1322	1.1797	1.2343	1.2960	1.3644
0.84	1.0037	1.0148	1.0334	1.0593	1.0926	1.1333	1.1813	1.2365	1.2989	1.3685
0.88	1.0037	1.0149	1.0335	1.0596	1.0931	1.1341	1.1824	1.2381	1.3011	1.3715
0.92	1.0037	1.0150	1.0337	1.0598	1.0935	1.1346	1.1832	1.2392	1.3026	1.3735
0.95	1.0037	1.0150	1.0337	1.0599	1.0937	1.1349	1.1835	1.2397	1.3033	1.3744
0.97	1.0037	1.0150	1.0337	1.0600	1.0937	1.1349	1.1837	1.2399	1.3036	1.3748
0.99	1.0037	1.0150	1.0337	1.0600	1.0937	1.1350	1.1837	1.2400	1.3037	1.3750
$\phi \rightarrow$	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	
0.80	1.4397	1.5212	1.6088	1.7017	1.7991	1.8993	1.9988	2.0895	2.1413	
0.84	1.4450	1.5284	1.6183	1.7144	1.8160	1.9222	2.0306	2.1360	2.2182	
0.88	1.4490	1.5337	1.6253	1.7238	1.8286	1.9393	2.0547	2.1715	2.2784	
0.92	1.4517	1.5373	1.6301	1.7302	1.8373	1.9511	2.0712	2.1962	2.3210	
0.95	1.4530	1.5390	1.6324	1.7332	1.8413	1.9566	2.0790	2.2078	2.3412	
0.97	1.4535	1.5396	1.6333	1.7344	1.8429	1.9588	2.0821	2.2125	2.3493	
0.99	1.4537	1.5400	1.6337	1.7349	1.8437	1.9599	2.0836	2.2147	2.3533	

notation, we see from Figure 2 that

$$\kappa R = c/\sinh\alpha \tag{7}$$

$$R = c/\sinh\beta \tag{8}$$

$$\epsilon = c(\coth\beta - \coth\alpha) \tag{9}$$

where ϵ is the distance between the centers of the inner and outer circles, that is, the eccentricity. A better measure of absolute eccentricity is the ratio of eccentricity to clearance or eccentricity ratio ϕ, which is a dimensionless quantity defined as

$$\phi = \epsilon/[R(1-\kappa)] \tag{10}$$

where κ is the ratio of the inner radius to the outer radius. From Equations (7) to (9), the constants c, α, and β are given in terms of κ and ϕ by the expressions

$$\cosh\alpha = [(1+\kappa) - (1-\kappa)\phi^2]/(2\kappa\phi) \tag{11}$$

$$\cosh\beta = [(1+\kappa) + (1-\kappa)\phi^2]/(2\phi) \tag{12}$$

$$c = R\sinh\beta = \kappa R\sinh\alpha \tag{13}$$

DEVELOPMENT OF THEORETICAL EQUATIONS

The method used to develop the falling cylinder viscometric equations consists of combining a force balance on the body, the equations of motion and continuity for the fluid, and a mass balance relating displaced fluid to that flowing through the annular gap between cylinder and tube, with the pertinent rheological model.

A force balance over the cylinder gives

$$(\rho_o - \rho)g\pi\kappa^2R^2 = (\Delta P/L)\pi\kappa^2R^2 + \langle\tau_w\rangle 2\pi\kappa R \tag{14}$$

simplifying

$$(\rho_o - \rho)g = (\Delta P/L) + 2\langle\tau_w\rangle/(\kappa R) \tag{15}$$

A mass balance gives

$$\pi\kappa^2R^2v_o = Q \tag{16}$$

The equation of motion is

$$\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = -(1/\mu)(\Delta P/L) \tag{17}$$

where

$$(\Delta P/L) = -\partial P/\partial z + \rho g \tag{18}$$

By the bipolar coordinate transformation of Equations (5) and (6) and the chain rule, Equation (17) becomes $\partial^2u/\partial\eta^2 + \partial^2u/\partial\xi^2 =$

$$-(1/\mu)(\Delta P/L)\,[c^2/(\cosh\eta - \cos\xi)^2] \tag{19}$$

This equation with the following boundary conditions [Equations (20) to (23)] can be solved by the method of separation of variables (for details, see reference 3):

$$(\partial u/\partial\xi)_{\xi=0} = 0 \tag{20}$$

$$(\partial u/\partial\xi)_{\xi=\pi} = 0 \tag{21}$$

$$u(\xi,\alpha) = -v_o \tag{22}$$

$$u(\xi,\beta) = 0 \tag{23}$$

The solution is

$$u = -(T/2)(\cosh\eta)/(\cosh\eta - \cos\xi) + E\eta + F + \sum_{n=1}^{\infty}(A_n e^{\eta n} + B_n e^{-\eta n})\cos n\xi \tag{24}$$

where

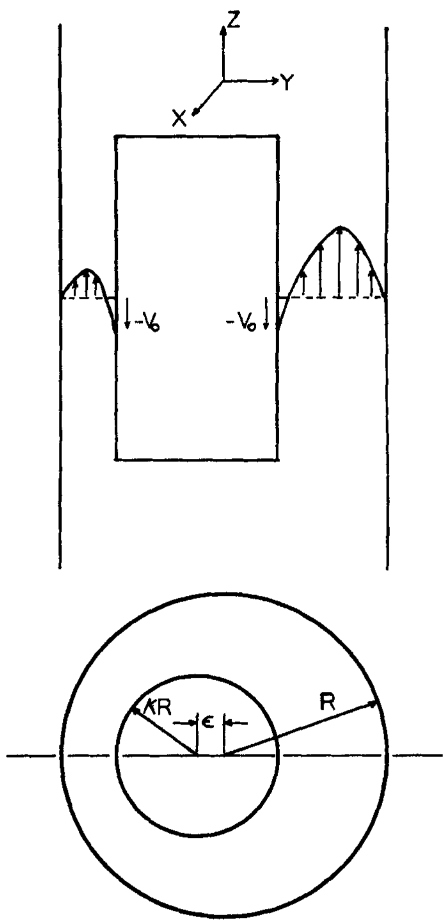

Fig. 1. Falling cylinder viscometer with eccentricity.

$$T = (c^2/\mu)(\Delta P/L) \tag{25}$$

$$E = T\,(\coth\beta - \coth\alpha)/[2\,(\beta-\alpha)] + v_o/(\beta-\alpha) \tag{26}$$

$$F = T\,(\beta\coth\alpha - \alpha\coth\beta)/[2(\beta-\alpha)] - (v_o\beta)/(\beta-\alpha) \tag{27}$$

$$A_n = T\,(\coth\beta - \coth\alpha)/(e^{2n\beta} - e^{2n\alpha}) \tag{28}$$

$$B_n = T(e^{2n\beta}\coth\alpha - e^{2n\alpha}\coth\beta)/(e^{2n\beta} - e^{2n\alpha}) \tag{29}$$

The fluid velocity equation, Equation (24), with coefficients expressed in Equations (25) to (29) is exact for laminar, incompressible, Newtonian fluid flow in eccentric annuli with the inner boundary moving at constant velocity $(-v_o)$; these equations reduce to the equations given by Snyder and Goldstein (*11*) for eccentric annuli with fixed walls.

The flux Q is obtained by integration of the fluid velocity distribution:

$$Q = \int\int u\,dx\,dy = 2\int_\alpha^\beta \int_0^\pi u|J|\,d\xi\,d\eta \tag{30}$$

where

$$J = \partial(x,y)/\partial(\xi,\eta) = \begin{vmatrix} \partial x/\partial\xi & \partial x/\partial\eta \\ \partial y/\partial\xi & \partial y/\partial\eta \end{vmatrix} = c^2/(\cosh\eta - \cos\xi)^2 \tag{31}$$

Substitution of Equations (24) and (31) into Equation (30) gives

$$Q = 2c^2 \int_\alpha^\beta \int_0^\pi \left\{ -(T/2)(\cosh\eta)/(\cosh\eta - \cos\xi) + E\eta + F + \sum_{n=1}^{\infty} (A_n e^{n\eta} + B_n e^{-n\eta})\cos n\xi \right\} \frac{d\xi\,d\eta}{(\cosh\eta - \cos\xi)^2} \tag{32}$$

Integrating Equation (32) and applying Equations (25) to (29) and Equations (7) to (9), one has the following expression for Q:

$$Q = (\pi/8\mu)(\Delta P/L)\left\{ R^4(1-\kappa^4) + 4c^2\epsilon^2/(\beta-\alpha) + 8c^2\epsilon^2 \sum_{n=1}^{\infty} \frac{ne^{-n(\beta+\alpha)}}{\sinh n(\beta-\alpha)} \right\} + \pi v_o\{(c\epsilon)/(\beta-\alpha) + (\kappa R)^2\} \tag{33}$$

This flux equation also reduces to the equation given by Piercy et al. (*10*) for the case where the walls are fixed.

The average inner shear stress $<\tau_w>$ is obtained by dividing the resistance of the core per unit length by the arc length of the core. It has been mentioned previously that τ_w is not constant angularly and thus tends to set a force couple. It is assumed in this development that the cylinder is restrained by pins or fins so that the couple cannot act.

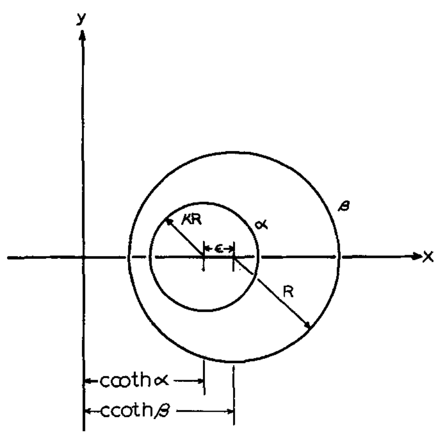

Fig. 2. Bipolar coordinates.

Table 2. Correction Factor Ψ_e' as a Function of Eccentricity Ratio ϕ and the Power Law Parameter n

ϕ	$n = 1$	$n = 0.5$	$n = 0.33$	$n = 0.25$
0.05	1.0038	1.007	1.013	1.019
0.10	1.0150	1.030	1.050	1.076
0.15	1.0337	1.067	1.114	1.172
0.20	1.0600	1.121	1.203	1.309

$$<\tau_w> = \left\{ -2\mu \int_0^\pi (\partial u/\partial\eta)_{\eta=\alpha}\,d\xi \right\} \Big/ (2\pi\kappa R) \tag{34}$$

Combination of Equation (34) with Equations (24) to (29) gives

$$-<\tau_w> = (\Delta P/L)(c\epsilon)/[2\kappa R(\beta-\alpha)] + (\mu v_o)/[\kappa R(\beta-\alpha)] + (\Delta P/L)(\kappa R/2) \tag{35}$$

Combining Equations (15), (16), (33), and (35) to eliminate $<\tau_w>$, $(\Delta P/L)$, and Q yields

$$(\mu v_o)/[(\rho_o - \rho)g] = (1/2)(\kappa R)^2(\beta-\alpha)/\{1/[(\beta-\alpha)S] - 1\} \tag{36}$$

where

$$S = (1-\kappa^4)/[4\phi^2(1-\kappa)^2\sinh^2\beta] + 1/(\beta-\alpha) + 4\sum_{n=1}^{\infty} \frac{n}{e^{2n\beta} - e^{2n\alpha}} \tag{37}$$

Equation (36) is the final equation relating terminal velocity of the falling cylinder to the density difference for a Newtonian fluid in the eccentric case.

To effect a reduction to the concentric case, one lets $\epsilon \to 0$, whence

$$\phi \to 0,\ c \to \infty,\ \beta \to \infty,\ \alpha \to \infty \tag{38}$$

but

$$(\beta - \alpha) \to \ln\kappa \tag{39}$$

$$4\phi^2\sinh^2\beta \to (1-\kappa^2) \tag{40}$$

$$S \to (1+\kappa^2)/(1-\kappa^2) + \ln(1/\kappa) \tag{41}$$

Substituting Equations (38) to (41) into Equation (36) gives

$$(\mu v_o)/[(\rho_o-\rho)g] = (1/2)(\kappa R)^2[\ln(1/\kappa) + (\kappa^2-1)/(\kappa^2+1)] \tag{42}$$

which is the equation given by Lohrenz et al. (*9*) for the concentric case.

From Equations (36) and (42), the terminal velocity ratio becomes

$$\Psi_e = (v_o)_{\text{ecc.}}/(v_o)_{\text{conc.}} = \frac{(\beta-\alpha)/\{1/[(\beta-\alpha)S] - 1\}}{[\ln(1/\kappa) + (\kappa^2-1)/(\kappa^2+1)]} \tag{43}$$

where α, β, and S are functions of κ and ϕ, as expressed by Equations (11), (12), and (37), respectively. Thus, the terminal velocity ratio Ψ_e is a function only of the two-dimensionless quantities κ and ϕ.

SOLUTION OF THE TERMINAL VELOCITY RATIO EQUATION

Solution of Equation (43) is quite straightforward. The sequence of solution is:

TABLE 3. COMPARISON OF Ψ_e' WITH Ψ_e FOR THE NEWTONIAN CASE

		$\phi = 0.05$	$\phi = 0.10$	$\phi = 0.15$	$\phi = 0.20$
Results From Equation (43)	$\kappa = 0.80$	1.0037	1.0147	1.0331	1.0589
	$\kappa = 0.86$	1.0037	1.0149	1.0335	1.0595
	$\kappa = 0.90$	1.0037	1.0149	1.0336	1.0598
	$\kappa = 0.95$	1.0037	1.0150	1.0337	1.0600
Results From Equation (46)		1.0038	1.0150	1.0337	1.0600

a. Solve Equations (11) and (12) for α and β for a given κ and ϕ

b. Solve Equation (37) for S

c. Solve Equation (43) for the terminal velocity ratio Ψ_e

Calculations were done on the IBM 7040/1401 computer with double precision for $0.80 \leq \kappa \leq 0.99$ and $0.05 \leq \phi \leq 0.95$. The series term $\sum_{n=1}^{\infty} n/(e^{2n\beta} - e^{2n\alpha})$ has been shown to converge rapidly (3).

Table 1 lists values of Ψ_e as a function of κ and ϕ. Figure 3 presents Ψ_e as a function of ϕ for $\kappa = 0.80$ and 0.99. It can be seen from Figure 3 that Ψ_e is almost independent of κ for $\phi < 0.6$. Table 1 can be used to set machine tolerances on a falling cylinder viscometer such that errors in terminal velocity due to eccentricity can be held within specified limits.

EXTENSION TO NON-NEWTONIAN FLUIDS

If the power law rheological model is used instead of the Newtonian model, the equation of motion, Equation (17), becomes

$$\partial(\partial u/\partial x)^n/\partial x + \partial(\partial u/\partial y)^n/\partial y = -(1/k)(\Delta P/L) \quad (44)$$

This equation is nonlinear and numerical solutions were not attempted in this work. However, in the following paragraphs we present an approximate method that was developed to estimate the effect of eccentricity in the flow of power law fluids in the falling cylinder viscometer (6).

The terminal velocity v_o of the cylinder in a falling cylinder viscometer, oriented either concentrically or eccentrically, is related to the average velocity in the annular gap by

$$v_o = [(1-\kappa^2)/\kappa^2] \langle v_z \rangle \quad (45)$$

$[(1-\kappa^2)/\kappa^2]$ is the ratio of the cross-sectional area available to flow in the annular space between cylinder and tube to the cross-sectional area of the cylinder and, regardless of the orientation of the cylinder with respect to the tube, this quantity is a constant for a given viscometer. Thus Ψ_e, defined by Equation (43), is also the ratio of $\langle v_z \rangle_{ecc.}$ to $\langle v_z \rangle_{conc.}$ for a given viscometer.

Although there is no solution for the terminal velocity (or average velocity) of the falling cylinder in a power law fluid, Vaughn (12) has solved the analogous problem in an eccentric annulus. Vaughn's solution gives

$$\Psi_e' = \langle v_z \rangle_{ecc.}/\langle v_z \rangle_{conc.} = [P_m(Z)]/Z^m \quad (46)$$

where $P_m(Z)$ are Legendre's polynomials of the first kind, and Z and m are defined as

$$\phi = (Z^2 - 1)^{1/2}/Z \quad (47)$$

and

$$m = (1 + 2n)/n \quad (48)$$

Table 2 gives values of Ψ_e' as a function of ϕ and n.

We now compare Ψ_e' as computed from Equation (46) with Ψ_e as computed from Equation (43) for Newtonian fluids ($n = 1$) in Table 3. Although Equation (46) shows Ψ_e' to be a function only of ϕ for a Newtonian fluid, we have demonstrated previously that Ψ_e should be a function of both ϕ and κ. However, the comparison in Table 3 shows that the agreement between Ψ_e' and Ψ_e is within two parts in 1,000 over the range $0.80 \leq \kappa \leq 1$, $0 \leq \phi \leq 0.20$. Thus we conclude that in this range, $\Psi_e' = \Psi_e$; that is, the eccentricity ratio is independent of the motion of one boundary of the falling cylinder viscometer relative to the other in Newtonian flow.

Since Ψ_e' is given as a function of n, a measure of the degree of non-Newtonian behavior in time-independent fluids, we suggest that Ψ_e' may be used to estimate the effect of eccentricity on the terminal velocity of a falling cylinder viscometer in such fluids. As the basis for our suggestion we cite the work of Ashare et al. (1), who used a similar approach by approximating the velocity of a concentric falling cylinder in a time-independent fluid from the average velocity in a slit with appropriate independent multiplicative corrections for the effects of the moving boundary and curvature. The validity of this method was subsequently established by Eichstadt and Swift (4), who showed that the error in approximating the terminal velocity of the cylinder in this way was less than 1% over the range $0 \leq (1/n) \leq 3$, $0.9 \leq \kappa \leq 1$.

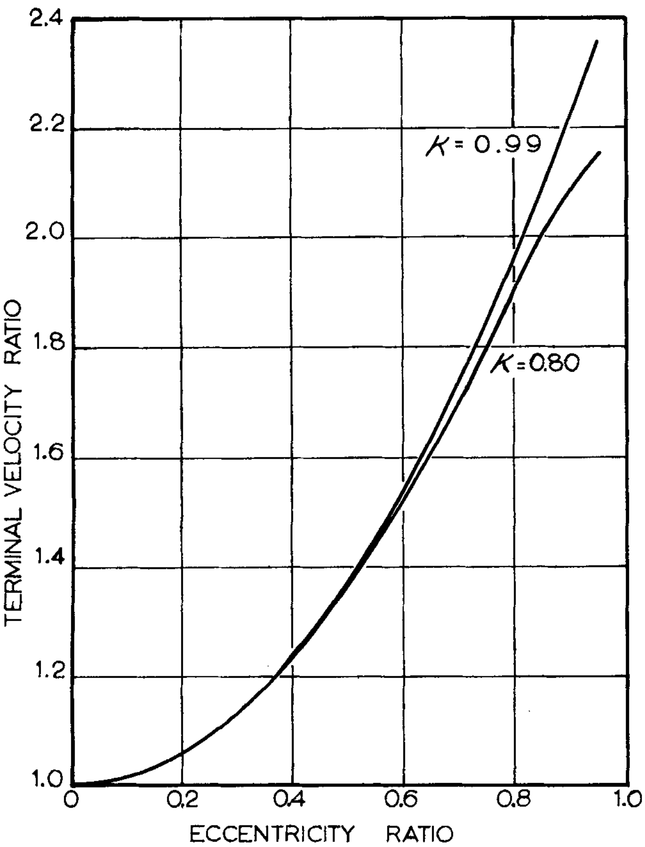

Fig. 3. Terminal velocity ratio vs. eccentricity ratio for $\kappa = 0.80$ and 0.99.

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NOTATION

A_n, B_n, E, F, T = constants as defined by Equation (24)
c = constant as defined by Equation (1), cm.
e = Napierian logarithm base, dimensionless
g = acceleration due to gravity, cm./sec.^{2}
i = $\sqrt{-1}$
J = Jacobian as defined by Equation (31)
k = power law model parameter, (dyne) (sec.^{n})/sq. cm.
m = variable as defined by Equation (48), dimensionless
n = power law model parameter, dimensionless
P = pressure, dynes/sq.cm.
$P_m(Z)$ = Legendre's polynomials of the first kind
Q = flux, flow rate, cc./sec.
R = inner radius of fall tube, cm.
S = constant as defined by Equation (37)
u = fluid velocity, cm./sec.
v_o = terminal velocity of falling cylinder, cm./sec.
$\langle v_z \rangle$ = average fluid velocity, cm./sec.
x, y, z = space coordinates, cm.
Z = variable as defined by Equation (47), dimensionless

Greek Letters

α, β = dimensionless radii of inner and outer surfaces of the falling cylinder viscometer (eccentric case)
$(\Delta P/L)$ = pressure gradient, dynes/cc.
ϵ = eccentricity, cm.
κ = ratio of cylinder radius to inner radius of fall tube, dimensionless
μ = Newtonian viscosity, g./(cm.) (sec.)
ξ, η = bipolar coordinates
ρ = fluid density, g./cc.
ρ_o = cylinder density, g./cc.
τ_w = shear stress at the wall, dynes/sq.cm.
$\langle \tau_w \rangle$ = average shear stress at the wall, dynes/sq.cm.
ϕ = eccentricity ratio, as defined by Equation (10), dimensionless
Ψ_e = terminal velocity ratio in falling cylinder viscometer, dimensionless
Ψ_e' = terminal velocity ratio in annulus, dimensionless

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Effect of Pressure Gradient on Sonic-Point Heat Transfer

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An analytical investigation of gas side heat transfer in high contraction, high convergence flows is presented. A partial criterion for retransition of a turbulent boundary layer is derived from the integral momentum equation and the effect of pressure gradient in the turbulent boundary-layer flow regime is described with the help of external flow relations. The validity of the analytical results is demonstrated by comparison to available internal and external flow, throat heat transfer data.

Accordingly, various turbulent boundary-layer heat transfer levels are obtained depending on the sonic-point pressure gradient, the specific heat ratio, the reference-to-stagnation temperature ratio, and the exponent of the viscosity-temperature function. Furthermore, the pressure gradient and the superficial mass velocity have opposite effects on the onset of retransition of a turbulent boundary layer.

In recent years there has been a considerable interest in compact combustors. Short combustion lengths offer substantial improvements in weight and cost. However, successful operation of such combustors requires knowledge of the heat transfer characteristics associated with the design aspects involved.

Compact combustors require relatively high flow-area contraction ratios, high convergence angles, and high wall curvatures at the throat, that is, the throat region is subjected to relatively high pressure gradients.

The effect of pressure gradient on heat transfer is complicated and diversified. In some range of Reynolds numbers and/or Mach numbers, increasing the pressure gradient tends to suppress the heat transfer level through the well observed phenomenon of reverse transition (*1* to *6*), while at the higher range of Reynolds numbers and/or